

Observations in the Einstein–de Sitter Cosmology: Dust Statistics and Limits of Apparent Homogeneity

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ABSTRACT

The two-point correlation function for the dust distribution in the unperturbed Einstein–de Sitter cosmological model is studied along the past light cone. It was found that the two-point correlation function seems unable to represent the theoretical distribution of dust along the backward null cone of this unperturbed model, which has already been determined in a previous paper as being apparently inhomogeneous at ranges usually considered as local. Such result was revisited in order to determine more precisely the quantitative limits where, in theory, we can detect apparent homogeneity, and it was found that this may only happen up to $z \sim 10^{-2}$. A different statistical analysis proposed by Pietronero is used, and it appears to be able to represent more accurately the theoretical distribution of dust in this cosmology. In the light of these results, it is argued that the usual practice of disregarding relativistic effects in studies of distribution of galaxies, by considering them as being placed on local regions, seems to be valid only on much closer scales than it is commonly believed, if the Einstein–de Sitter model is used as a theoretical framework for studying such distributions. In this cosmology with $H_0 = 75 \text{ Km s}^{-1} \text{ Mpc}^{-1}$, that may only happen in redshifts as low as $z \approx 0.04$, which means that the local approximation seems to be valid up to zeroth order of approximation only. As at present there are many redshift surveys which have already probed at deeper ranges, it seems that in order to compare the Friedmann models with observations we have to be very careful when ignoring the past light cone problem in observational cosmology, either in theoretical calculations or in data analysis, due to relativistic effects which produce observable inhomogeneity even in spatially homogeneous cosmological models.

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1 Introduction

The spatial two-point correlation function is perhaps the most popular statistical tool currently used for the characterization of the large-scale distribution of galaxies. It is extensively used to extract basic statistical information from galaxy catalogues, measuring their clustering properties, information which is then often used by theoreticians in their cosmological models. The most important result seems to be the finding that at small scales the spatial two-point correlation function is observed to follow a power law with negative slope (Peebles 1980), and as the correlation function decreases towards one or zero when the distance increases, the distance where it reaches these values is usually associated with the homogeneization of the sample in the sense that we are supposed to have reached a homogeneous “fair sample” of the universe (Peebles 1980).

Nevertheless, doubts about the reliability of the correlation function have been voiced by some authors. Einasto, Klypin & Saar (1986) and Davis et al. (1988) found out that the correlation length r_0 , which is supposed to indicate the depth where we have reached a “fair sample” of the distribution of galaxies, seems to increase with the sample size (see also Martínez et al. 1993). Pietronero (1987) criticized the use of the correlation function in the context of the distribution of the galaxies under the grounds that it is conceptually incorrect since it could only be applied to physical systems whose average densities are well defined quantities, property, his argument goes, apparently not found up to now in the observed distribution of galaxies (see also Coleman, Pietronero & Sanders 1988). Geller (1989) voiced a similar concern as her slices have inhomogeneities that are large compared with the sample volume. In addition, Coleman & Pietronero (1992) carried out simulations of some computer generated samples and found out that the spatial two-point correlation function seems to be unable to describe correctly the samples studied, whose distribution were known by construction. Such result is claimed by them to give support to the hypothesis that the observed large-scale distribution of galaxies does seem to follow a fractal pattern. Finally, by using a different from usual correlation analysis, Coleman, Pietronero & Sanders (1988) reported that there is no indication so far of any correlation length in the CfA samples, that is, no indication of a homogeneous “fair sample” being reached, which seems to contradict the results obtained from the measurements of the usual correlation function where a homogeneization of the distribution of galaxies is indicated.

Another important aspect that remains basically untested is the question of at what scales the correlation function itself may be affected by the curvature and expansion of the universe, and the answer to this question might impose important limits in its usefulness. In particular, it is necessary to determine quantitatively the limits of observational detectability of the spatial homogeneity of the popular spatially homogeneous Friedmann models, since observations are carried out along our past light cone and what we can directly observe are galaxies placed at different distances at different times. This last remark is of special importance since Ribeiro (1992) showed that if the universe is really Einstein-de Sitter, departures

from local apparent homogeneity should be observed at much closer scales than it is usually assumed.

This paper attempts to address the issues surrounding this controversy from a different perspective than the authors mentioned above. By starting from the most popular relativistic cosmological model, that is, the Einstein-de Sitter model, I study its unperturbed behaviour along the past light cone and obtain some theoretical predictions which can be used to analyse results obtained through actual astronomical observations. Therefore, I do not make the usual assumption that relativistic effects can be ignored and Euclidean geometry can be used when dealing with the typical depths of the current redshift surveys. On the contrary, I intend to ascertain to what extent this hypothesis may be really a good approximation.

The aim of this paper is twofold. First to further analyse the basic result of Ribeiro (1992), which, simply speaking, states that relativistic corrections do matter in cosmology, even at small scales. In special, one of the main goals here is to study the relativistic effects on some statistical tests used in the study of large-scale distribution of galaxies. Second, to attempt to set up explicit quantitative limits to the observation of spatial homogeneity in the unperturbed Einstein-de Sitter model, the ranges of observational detectability of the homogeneous hypothesis, relating those limits to the error margins associated with astronomical measurements, and trying to see which statistical test best reproduces the actual distribution of dust in this model.

This paper is organized as follows. In §2 I present a brief summary of the basic observational relations necessary for the model under study, which were obtained in Ribeiro (1992), and in §3 I derive the correlation function and other functions along the past light cone. Section 4 shows an analysis of the results obtained in the previous section, presents some limits to the observation of apparent homogeneity in the unperturbed Einstein-de Sitter model, and discuss how the correlation function seems to be an inadequate tool to characterize the distribution of galaxies, while the statistical analysis advanced by Pietronero (1987) appears to describe more accurately the actual distribution of dust in this model. The paper finishes with a discussion on the various possible implications of these results.

2 Observational Relations in the Einstein-de Sitter Space-time

Let us write the Einstein-de Sitter metric (with $c = G = 1$, $\Lambda = 0$) in the form

$$dS^2 = dt^2 - a^2(t) \left[dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \quad (1)$$

where the function $a(t)$ is given by

$$\left(\frac{da}{dt} \right)^2 = \frac{8\pi}{3} \rho a^2 \quad (2)$$

and the local density is

$$\rho = \frac{1}{6\pi a^3(t)}. \quad (3)$$

If we label $t = 0$ as “now”, that is, $t = 0$ being defined as our present time hypersurface, the solution of equation (2) may be written as

$$a(t) = \left(t + \frac{2}{3H_0}\right)^{2/3}, \quad (4)$$

where H_0 is the present value of the Hubble constant.

We are interested in studying this metric along the past light cone inasmuch as this hypersurface is where astronomical observations are actually made, and for this purpose it becomes necessary to integrate the past radial null geodesic of metric (1),

$$\frac{dt}{dr} = -a(t),$$

from “here and now” ($t = r = 0$) till $t(r)$. The result may be written as

$$3\left(t + \frac{2}{3H_0}\right)^{1/3} = \left(\frac{18}{H_0}\right)^{1/3} - r. \quad (5)$$

Here I have chosen to use the radius coordinate r as the parameter along the null geodesic.

The observational relations necessary in this paper have already been calculated in Ribeiro (1992). Along the backward null cone the cumulative number count N_c , the luminosity distance d_ℓ , the redshift z , the observed volume V and the average density $\langle\rho\rangle$ are respectively given by

$$N_c = \frac{2r^3}{9M_G}, \quad (6)$$

$$d_\ell = 9r\left(\frac{2}{3H_0}\right)^{4/3}\left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^{-2}, \quad (7)$$

$$1 + z = \left(\frac{18}{H_0}\right)^{2/3}\left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^{-2}, \quad (8)$$

$$V \equiv \frac{4}{3}\pi d_\ell^3 = \frac{192\pi r^3}{H_0^4\left[(18/H_0)^{1/3} - r\right]^6}, \quad (9)$$

$$\langle\rho\rangle \equiv \frac{N_c M_G}{V} = \frac{H_0^4}{864\pi}\left[\left(\frac{18}{H_0}\right)^{1/3} - r\right]^6, \quad (10)$$

where M_G is the average galactic rest mass ($\sim 10^{11}M_\odot$).

If we use the inverse of equation (7), the cumulative number counting and the average density may be written in terms of the luminosity distance as

$$N_c = \frac{4}{H_0 M_G} \left[1 - \left(\frac{1}{2} + \sqrt{\frac{H_0 d_\ell}{2} + \frac{1}{4}}\right)^{-1}\right]^3, \quad (11)$$

$$\langle \rho \rangle = \frac{3H_0^2}{8\pi} \left(\frac{1}{2} + \sqrt{\frac{H_0 d_\ell}{2} + \frac{1}{4}} \right)^{-6}. \quad (12)$$

Before closing this section, a few words are necessary here in order to explain why the luminosity distance d_ℓ was chosen as the measurement of distance in this paper. As is well known, in relativistic cosmology we do not have an unique way of measuring the distance between source and observer since such measurement depends on circumstances. We can, for instance, make use of geometrically defined distances like the proper radius, or observationally defined distances like the luminosity distance or the observer area distance (also known as angular diameter distance) in order to say that a certain object lies at a certain distance from us. The circumstances which tell us which definition to use can also be determined on observational grounds, and so if we only have at our disposal the apparent magnitudes of galaxies we associate to each of them the luminosity distance and use such measurement in our analysis. On the other hand, if these apparent magnitudes are corrected by the redshift of the sources, then we can associate the corrected luminosity distance, which is the same as the observer area distance obtained if we have the apparent size of the objects (Ellis 1971), and, therefore, another kind of distance measure is obtained. Any of these observational distances is as valid as any other, as real as any other, with the choice being dictated by the availability of data, the nature of the problem being treated and its convenience, but they will only have the same value at $z \ll 1$, varying, sometimes widely, for larger z (see McVittie 1974 for a comparison of these distances in simple cosmological models).

In this paper we are interested in observables because we seek to compare theory with observations, and this means that geometrical distances are of no interest here. Consequently, the approach of this paper is different from others where unobservable coordinate distances (differences between coordinates) and separations (integration of the line element dS over some previously defined surface) are taken as measure of distance, and in order to develop a treatment coherent with the observational approach of this problem we need to make a choice among the observational distances based on the nature of the problem and the observations available.

In this work I intend to study the theoretical distribution of density and the correlation function obtained in the Einstein-de Sitter model in order to compare their forms with the ones produced by the recent all-sky redshift surveys, and in this field it is usual for observers to take the luminosity distance as their indicator of distance (for example, Saunders et al. 1990 do use d_ℓ in their extensive statistical analysis of a sample of IRAS galaxies). It seems therefore perfectly reasonable to take the luminosity distance as the most appropriate definition of distance to use in the context of this work, because what is sought is to mimic the current methodology used by many observers in this field, and to carry out a comparison between the theoretical predictions of this model and the observational results brought by the redshift surveys.

3 Dust Statistics Along the Past Light Cone

Once we have established the basic observational relations which shall be needed here, the next step is to use those relations in order to find the expression of the correlation function for the distribution of dust in the Einstein-de Sitter model. Nonetheless, before we go into the details of the calculations themselves, a few remarks about the physical behaviour of the model under study will be helpful in order to make clear the meaning of the results obtained.

The Einstein-de Sitter model is spatially homogeneous, and this is obvious from equation (3) since each specific value of the time coordinate corresponds to another specific and constant value for the local density. Therefore, at hypersurfaces of constant time, that is, at spatial sections of the model, the density remains unchanged, and at our present time hypersurface the local density is given by the constant

$$\rho_0 = \frac{3H_0^2}{8\pi}. \quad (13)$$

It is well known that this result cannot be accepted at its face value since astronomical observations indicate lumpiness of matter, and a constant local density would mean that there would not be galaxies, at least at smaller scales. As a consequence of this, the current view is to assume that there are metric perturbations such that local density fluctuations $\delta\rho/\rho$ arise and produce the observed lumpiness of matter. Then the current thinking assumes that beyond certain length scale the density fluctuations will decrease, and eventually approach asymptotically the value ρ_0 . Such reasoning naturally leads to a statistical interpretation of equation (13) in the sense that ρ_0 would be the average value of the local density fluctuations, and, under this view, the cosmological problem becomes the determination of the value of ρ_0 and the density perturbations scale length. This discussion is also valid for the other Friedmann models, either open or closed, with one of the differences being the value of ρ_0 , but in this paper I shall only deal with the Einstein-de Sitter model, for reasons of simplicity and analytical feasibility.

The spatial two-point correlation function is the most common statistical tool currently used with the aim of studying the cosmological density fluctuations. It assumes that the objects under discussion (galaxies) can be regarded as point particles that are distributed homogeneously on sufficiently large scale. This means that we can meaningfully assign an average number density to the distribution and, therefore, we can characterize the galaxy distribution in terms of the extent of the departures from uniformity on various scales. Consequently, the correlation function would, in principle, give us a methodology by which one hopes to achieve both aims as described above: to give the approximate density perturbation scale length and, indirectly, the value of ρ_0 (not necessarily the specific one given by the Einstein-de Sitter model, eq. [13]), and that would be indicated at the ranges where the correlation function would tend to zero, meaning the approach to a Poisson distribution and, therefore, the expected homogeneity at deeper scales. Once the range of homogeneity is

established we would have achieved a “fair sample” of the Universe and we could, in theory, calculate ρ_0 by means of, say, counts of galaxies beyond the fair sample range.

Such scheme, however, usually disregards an essential point: observations are carried out along the past light cone and, hence, relativistic corrections might become important at larger scales. In particular, inasmuch as along the past light cone the proper density changes due to the crossing of the null geodesic through hypersurfaces of constant time, but with different values of the proper density, a cumulative average density will change for the same reason and, therefore, we will end up having a not well defined average number density for the distribution of dust even in Friedmann cosmologies (Ribeiro 1992, 1993). In other words, since the past light cone is a hypersurface of inhomogeneity (the density varies along it), as far as observations are concerned, in the spatially homogeneous Friedmann models their geometrical homogeneity is in fact a local apparent homogeneity, which becomes an apparent inhomogeneity for deeper observational ranges. This physical property of the models is related to the fact that astronomical observations are made along the special null hypersurface which is in fact inhomogeneous. There is no contradiction in the fact that a spatially homogeneous model does have hypersurfaces of inhomogeneity, as it continues to exhibit the physical property of having constant density at constant time slices. Therefore, it is important to state clearly that the Einstein-de Sitter model is not only *spatially homogeneous*, but also *apparently inhomogeneous*. This effect of observational inhomogeneity of the model may also be termed as “evolution of the background”.

The implicit answer to the points raised above is to argue that at small scales, that is, at small redshifts, relativistic effects are not important, and so we can carry out calculations and analysis of data using relations valid only along our present time hypersurface, and ignore the null geodesic problem because the values of z of the all-sky redshift surveys are supposed to be small, and would produce insignificant relativistic corrections in the observables. In other words, since at small scales the apparent inhomogeneity turns out to be local apparent homogeneity, then, the reasoning goes, the null geodesic could be ignored. Nevertheless, putting the problem in that way the key question remains hidden: how small should the scales be such that we can ignore the backward null cone? In other words, we need a criterion to tell us *quantitatively* to what extent we can still consider small scales as small such that we could then safely disregard the past light cone problem up to that range, or still putting the problem differently, we need to know the scales where we would no longer be receiving photons from objects located at our own present time hypersurface (Ellis 1987, p. 62; MacCallum 1987, p. 135). The correlation function does not help to answer this question as it already *assumes* that homogeneity will be reached at certain scale, and so it is unable to test the local homogeneous hypothesis itself (see Pietronero 1987, and Coleman & Pietronero 1992 for a full discussion). This is an essential point because it has already been indicated in a previous paper (Ribeiro 1992) that such a departure from our local region may happen at redshifts as low as $z \approx 0.04$, value which is usually considered as well within our local region,

and in that case the implications for the measurement of the galaxy and cluster correlation functions may be important.² Equation (12) shows clearly that the Einstein-de Sitter model does not seem to remain homogeneous along the past light cone (see Ribeiro 1992 for a more detailed physical discussion about this behaviour, and Ribeiro 1993 for its extension to the other Friedmann models), and as said above, once the average density stops having a well defined value, beyond a certain length scale, we can no longer assign a well defined average number density to the distribution of galaxies, which means a break down of the usual correlation function. Bearing this discussion in mind we can now proceed to determine the expression of this function for the distribution of dust in the unperturbed Einstein-de Sitter model.

I shall approach the correlation function $\xi(d_\ell)$ through the “conditional density” $\Gamma(d_\ell)$, which is simpler to calculate in this context. According to Pietronero (1987; see also Coleman & Pietronero 1992), $\Gamma(d_\ell)$ is defined by

$$\Gamma(d_\ell) = \frac{1}{S} \frac{dN_c}{d(d_\ell)}, \quad (14)$$

where $S(d_\ell) \equiv 4\pi d_\ell^2$ is the area of the observed spherical shell of radius d_ℓ . This equation shows that the conditional density actually measures the average density at a distance d_ℓ from an occupied point.³ Taking into account equation (11), the conditional density for the distribution of dust in the Einstein-de Sitter model yields

$$\begin{aligned} \Gamma(d_\ell) = & \left(\frac{3H_0^2}{\pi M_G \sqrt{1 + 2H_0 d_\ell}} \right) \left[(1 + H_0 d_\ell) (4 + 8H_0 d_\ell + H_0^2 d_\ell^2) + \right. \\ & \left. + (2 + H_0 d_\ell) (2 + 3H_0 d_\ell) \sqrt{1 + 2H_0 d_\ell} \right]^{-1}. \end{aligned} \quad (15)$$

We can also define a “conditional average density” (Pietronero 1987; Coleman & Pietronero 1992)

$$\Gamma^*(d_\ell) = \frac{1}{V} \int_V \Gamma(d_\ell) dV = \frac{3}{d_\ell^3} \int_0^{d_\ell} x^2 \Gamma(x) dx, \quad (16)$$

which gives the behaviour of the average density of a sphere of radius d_ℓ centered around an occupied point averaged over all occupied points. By means of equations (15) and (16) the conditional average density is found to be

$$\begin{aligned} \Gamma^*(d_\ell) = & \frac{3}{d_\ell^3} \lim_{b \rightarrow 0} \left\{ \frac{1}{\pi M_G b^3 d_\ell^3 H_0^4} \left[(2 + 3bH_0) (2 + bH_0) d_\ell^3 \sqrt{1 + 2bH_0} + \right. \right. \\ & + (2 + 3H_0 d_\ell)^2 b^3 - \sqrt{1 + 2H_0 d_\ell} (2 + 3H_0 d_\ell) (2 + H_0 d_\ell) b^3 - \\ & \left. \left. - 9b^2 H_0^2 d_\ell^3 - 12bH_0 d_\ell^3 - 4d_\ell^3 \right] \right\}, \end{aligned}$$

² By local region I mean the region along the past light cone where observational relations evaluated at t = now are still a valid approximation when data error margins are considered.

³ Actually, some of Pietronero’s (1987) treatment was anticipated by Wertz (1970, p. 43-45), where the definition of a “conditional density” appears under the name “differential density” (Wertz 1970, p. 17; see also Wertz 1971).

and once this limit is performed we get

$$\Gamma^*(d_\ell) = \left(\frac{3}{\pi M_G H_0^4 d_\ell^6} \right) \left[(1 + H_0 d_\ell) (4 + 8H_0 d_\ell + H_0^2 d_\ell^2) - (2 + H_0 d_\ell) (2 + 3H_0 d_\ell) \sqrt{1 + 2H_0 d_\ell} \right]. \quad (17)$$

After some algebraic manipulation we find that in the limit when $d_\ell \rightarrow 0$, $\Gamma^* \rightarrow (3H_0^2/8\pi M_G)$, as one expects.

As shown by Pietronero (1987; see also Martínez et al. 1993), the correlation function $\xi(d_\ell)$ is related to $\Gamma(d_\ell)$ in a simple way:

$$\xi = \frac{\Gamma}{\langle n \rangle} - 1, \quad (18)$$

where $\langle n \rangle \equiv \langle \rho \rangle / M_G$ is the average number density. For the model under study we have

$$\langle n \rangle = \frac{3H_0^2}{8\pi M_G} \left(\frac{1}{2} + \sqrt{\frac{H_0 d_\ell}{2} + \frac{1}{4}} \right)^{-6}. \quad (19)$$

One can verify that equations (17) and (19) are equal, which comes as no surprise since, by its own definition, Γ^* indeed describes the average number density. Therefore, we may write the identity

$$\Gamma^* = \langle n \rangle. \quad (20)$$

This result is in agreement with the point already made by Coleman & Pietronero that “for a test of general tendencies, like homogeneity versus power law correlations, this (Γ^* function) is by far the best test” since “it correctly reproduces global properties which are the object of the present discussion” (Coleman & Pietronero 1992, p. 328). In fact, inasmuch as the average number density $\langle n \rangle$ gives us the form of the distribution of dust along the backward null cone in the Einstein-de Sitter model, equation (20) is actually telling us that the conditional average density Γ^* is an appropriate function to be used in galaxy catalogues in order to see if the large-scale distribution of galaxies does follow a dust pattern as given by $\langle n \rangle$ in the Einstein-de Sitter cosmological model.

It is worth writing the power series expansion of equation (19),

$$\langle n \rangle = \frac{3H_0^2}{8\pi M_G} \left(1 - 3H_0 d_\ell + \frac{27}{4} H_0^2 d_\ell^2 - \frac{55}{4} H_0^3 d_\ell^3 + \dots \right), \quad (21)$$

and this expression already shows very clearly that the overall number density is only constant at very small luminosity distances, that is, at the zeroth order of approximation. In addition, its zeroth order term is equal to equation (13), apart from a constant, as it should be.

In order to use equation (19) to evaluate the correlation function, we have to suppose that there are m objects confined to a fixed region R of volume V_u (Peebles 1980, p. 145). Hence, the expression for the two-point spatial correlation function for the dust distribution in the Einstein-de Sitter model, and along the past light cone, is given by

$$\xi(d_\ell) = \left[\left(R^3 H_0^2 + 9R^2 H_0 + 12R - 6H_0^2 d_\ell^3 - 19H_0 d_\ell^2 - 16d_\ell \right) H_0 \sqrt{1 + 2H_0 d_\ell} + \right.$$

$$\begin{aligned}
& + (2 + RH_0) (2 + 3RH_0) \sqrt{(1 + 2H_0d_\ell) (1 + 2RH_0)} - (1 + H_0d_\ell) (1 + 2H_0d_\ell) \times \\
& \times \left(4 + 8H_0d_\ell + H_0^2d_\ell^2 \right) \left\{ (1 + 2H_0d_\ell) [(2 + H_0d_\ell) (2 + 3H_0d_\ell) \times \right. \\
& \left. \times \sqrt{1 + 2H_0d_\ell} + (1 + H_0d_\ell) (4 + 8H_0d_\ell + H_0^2d_\ell^2) \right\}^{-1}, \tag{22}
\end{aligned}$$

where the dependence on the sample size R is explicit.

As a final remark before the end of this section, it is important to stress the explicit dependence of all functions with the luminosity distance, an effect which comes from the fact that along the past light cone the spatially homogeneous Einstein-de Sitter model is apparently inhomogeneous, because the proper density changes since equation (3) is a function of time (Ribeiro 1992). Therefore, it is essential to determine quantitatively the length scale where the apparent inhomogeneities start to play a significant role in the observational quantities above.

4 Analysis

As discussed in the previous section, it is usually assumed that at small redshifts the backward null cone can be ignored and Euclidean geometry may be used as a good approximation in the study of distribution of galaxies (see e.g. Peebles 1980, p. 143). Such reasoning seems perfectly reasonable once the linear redshift-distance law represents well the observations at small redshifts. Indeed, the redshift-distance relation along the past light cone in the unperturbed Einstein-de Sitter model is easily obtainable from equations (7) and (8) as being

$$d_\ell = \frac{2}{H_0} \left(1 + z - \sqrt{1 + z} \right), \tag{23}$$

whose power series expansion reads

$$d_\ell = \frac{z}{H_0} + \frac{z^2}{4H_0} - \frac{z^3}{8H_0} + \dots \tag{24}$$

Therefore, the Hubble law is clearly the first order approximation of equation (23), and if we take $H_0 = 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$ it is easy to see that at $z = 0.1$ the contribution of the second order term in equation (24) is about 2.4% of the total in equation (23).⁴ Even at $z = 0.5$ the second order term contributes to only around 11%, which means the linear approximation of equation (23) can be considered as valid up to at least $z \approx 0.5$ once error margins of the same magnitude are considered. This reasoning is perfectly standard and is being repeated here just for the sake of clarity of the results which will follow.

The surprising aspect of the analysis of the observational relations along the past null cone is the effect of small redshifts on the average number density. Considering equations

⁴ From now on I shall assume $H_0 = 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$ unless stated otherwise. Units are such that $c = G = 1$, and so distance is given in gigaparsecs (10^9 pc), mass in units of $2.09 \times 10^{22} M_\odot$ and time in units of 3.26 Gyr ($1\text{yr} = 3.16 \times 10^7 \text{ s}$).

(19) and (23) we can write $\langle n \rangle$ as a function of the redshift,

$$\langle n \rangle = \frac{3H_0^2}{8\pi M_G} \frac{1}{(1+z)^3}. \quad (25)$$

Figure 1 shows a plot of equation (25) and it is clear the overall density decreases quite sharply as z increases. A 10% drop in $\langle n \rangle$, from the value at our present time hypersurface, occurs at $z = 0.036$ (≈ 145 Mpc), which is within the limits of many redshift surveys. At $z = 0.07$ (≈ 285 Mpc, depth of IRAS galaxies), the density is around 80% from its value now ($t = 0$), and at $z = 0.1$ the density drops to 75% from its initial value. It is therefore clear that even accepting an error margin of 25% in the measurements of the global density, a redshift equals to 0.1 is approximately the deepest scale where we could observe a homogeneous distribution of dust in an Einstein-de Sitter model ($z = 0.1 \Rightarrow d_\ell \approx 410$ Mpc if $H_0 = 75 \text{ Km s}^{-1} \text{ Mpc}^{-1}$; for $H_0 = 100 \text{ Km s}^{-1} \text{ Mpc}^{-1}$, $d_\ell \approx 310$ Mpc). Note that at $z = 0.1$ the redshift-distance law is still well approximated by a linear function, with an error margin of less than 2.5%. At $z = 0.5$, the average number density $\langle n \rangle$ goes down to only 30% from its initial value at $t = 0$. Consequently, by this method, that is, by using the average density as a gauge, we are able to determine *quantitatively* the scales where the local apparent homogeneity still holds, and the few numbers above show very clearly that the end of the observable homogeneous region in the model under study may happen at scales much smaller than usually assumed, being not only already inside the limits of many redshift surveys, but also well within the region of validity of the Hubble law. Table 1 summarizes the numerical estimates given above.

Let us see now how the correlation function is affected when measured along the past light cone. The power series expansion of equation (22) is

$$\xi(d_\ell) = \frac{(\Xi - 4)}{8} - \frac{(\Xi + 4)}{2} H_0 d_\ell \left[1 - \frac{45}{16} H_0 d_\ell + \frac{55}{8} H_0^2 d_\ell^2 - \frac{1001}{64} H_0^3 d_\ell^3 + \dots \right], \quad (26)$$

where

$$\Xi(R) \equiv \sqrt{1 + 2RH_0} (2 + RH_0) (2 + 3RH_0) + R^3 H_0^3 + 9R^2 H_0^2 + 12RH_0. \quad (27)$$

As in the case of the average number density only the zeroth order term is constant. This is obviously caused by the fact that the spatially homogeneous Einstein-de Sitter model is only apparently homogeneous, and at very low observational distances. Note, however, that even this zeroth order term depends on the size of the sampling, as pointed out by Pietronero (1987), and has no resemblance to the observed form of the correlation function at very small distances, known to follow a power-law form (Peebles 1980). This last remark is hardly surprising since the unperturbed Einstein-de Sitter model does not model the small-scale lumpiness of matter, and therefore this unperturbed model cannot reproduce the observed correlation function. Nonetheless, at larger scales $\xi(d_\ell)$ seems to present some odd effects. Figure 2 shows the correlation function (22) plotted for different values of the sample size

R , and it is clear that even at the regions where the model is apparently homogeneous, as shown in figure 1, $\xi(d_\ell)$ seems unable to fully characterize it. With $R = 50$ Mpc ($z \approx 0.013$), $\xi(d_\ell)$ seems to indicate a breakdown of the homogeneous pattern starting around 10 Mpc ($z \approx 0.003$), which clearly contradicts equation (25) and its plot in figure 1, since both indicate a continuation of the local homogeneous region to at least 145 Mpc ($z \approx 0.036$), with a 10% error. For different values of R a similar situation happens, and although we have seen that equation (25) shows an unique and clear scale length where the observable homogeneity stops, the correlation function does not seem to be able to characterize such distance. For bigger values of R the results obtained from the correlation function become meaningless since according to equation (25) the average number density stops being a well defined quantity, and therefore, we can no longer make use of the correlation function in this context. Hence, the application of the correlation function in this context appears to be severely limited to only the linear region with zero slope of figure 1, with its limit being determined by the error margin considered.⁵ It is also interesting to note that the inflexion of $\xi(d_\ell)$ in figure 1 seems to be related to the value of R chosen in each plot, and this is consistent with Pietronero's (1987) critique of the correlation function in the sense that it mixes up the physical properties of the samples studied with their sizes. Hence, although the use of the correlation function in this unperturbed model does not relate the results to actual observations, it seems clear that the correlation function is a poor statistical test since if the Universe were *exactly* Einstein-de Sitter, the correlation function would have difficulties to characterize the global behaviour of the dust in the model.

In contrast to the correlation function, the conditional density $\Gamma(d_\ell)$ seems to be able to better trace the dust distribution in the Einstein-de Sitter model. To see this let us first carry out a power series expansion in equation (15). The result may be written as

$$\Gamma(d_\ell) = \frac{3H_0^2}{8\pi M_G} \left(1 - 4H_0 d_\ell + \frac{45}{4} H_0^2 d_\ell^2 - \frac{55}{2} H_0^3 d_\ell^3 + \dots \right), \quad (28)$$

where one can see that the zeroth order term is the local density at the present time (13), apart from a normalizing constant, and is the same factor appearing in equation (21). As in the case of the average number density $\langle n \rangle$, one can see in equation (28) that deviations from local homogeneity are a first order effect. Figure 3 shows a plot for equation (15) where it is clear the breakdown of the local apparent homogeneity as the luminosity distance increases, the same effect shown in figure 1.

However, in the sense of results that could be derived directly from galaxy catalogues, $\Gamma(d_\ell)$ does not seem to be as a good quantitative tracer for the behaviour of the average density $\langle n \rangle$ as the conditional average density $\Gamma^*(d_\ell)$ appears to be. This happens because $\Gamma^*(d_\ell)$ traces $\langle n \rangle$ exactly (eq. 20) while $\Gamma(d_\ell)$ does not, and hence $\Gamma^*(d_\ell)$ is certainly the function to be calculated from galaxy catalogues in order to probe the behaviour of the mean

⁵ For example, for a 10% error the correlation function can only be used up to $z \approx 0.04$. For an error around 25%, up to $z \approx 0.1$, etc.

number density $\langle n \rangle$. Table 2 presents the percentage drop of $\Gamma(d_\ell)$ and $\langle n \rangle$ with increasing redshifts when they are compared with their values at the present time hypersurface $t = 0$. Since $\langle n \rangle$ gives the actual distribution of dust of the model, table 2 shows that although $\Gamma(d_\ell)$ reproduces the general tendency for the drop in the density, it is not much sensitive to the drop itself. That seems to happen because, by its own definition, $\Gamma(d_\ell)$ is more affected by local fluctuations since it measures the average density at a distance d_ℓ from an occupied point, while $\Gamma^*(d_\ell)$ does this but also averages over all occupied points. Clearly the latter function is not appropriate when one is investigating local fluctuations because in that case these fluctuations would be smoothed out in the average. Nevertheless, Γ^* would be the desired test to be used when one is interested in global tendencies and properties, which is the case when one is studying the limits of the apparent homogeneity of the Friedmann cosmologies.

Despite those limitations, the conditional density can still be used in statistical analyses of the local region of the model. To see this, let us find the ratio between Γ^* and Γ . By using equations (15) and (17) one can show that

$$\frac{\Gamma^*}{\Gamma} = \sqrt{1 + 2H_0 d_\ell}. \quad (29)$$

By means of equation (23) we may write the equation above as a function of the redshift,

$$\frac{\Gamma^*}{\Gamma} = 2\sqrt{1+z} - 1, \quad (30)$$

whose power series expansion yields,

$$\frac{\Gamma^*}{\Gamma} = \left(1 + z - \frac{z^2}{4} + \frac{z^3}{8} - \frac{5}{64}z^4 + \dots \right). \quad (31)$$

It is therefore clear that the conditional average density Γ^* and the conditional density Γ are equal to each other at the zeroth order of approximation in the Einstein-de Sitter model, that is, at the Newtonian approximation. Hence, in our local region the two functions will basically give the same information about the distribution of density, and will start to differ outside our local homogeneous region. However, the ratio Γ^*/Γ does not provide a good quantitative indicator for the end of our local region since both Γ^* and Γ will change outside our neighbourhood, with their ratio being unable to be an accurate gauge of where such a departure from our local apparent homogeneity will occur.

5 Conclusion and Discussion

In this paper I have studied along the past light cone the two-point correlation function, and a different statistical analysis proposed by Pietronero (1987), for the dust distribution in the unperturbed Einstein-de Sitter cosmological model. By using observational relations already derived in Ribeiro (1992), I explicitly obtained the forms for the correlation function, the

conditional density, and the conditional average density in the backward null cone, where they are astronomically observable. By comparing the average number density at our local time hypersurface and its change along the past null geodesic, it was possible to obtain explicit quantitative limits for the observable local homogeneity in the model, and the results showed that the possible detection of apparent homogeneity are at lower redshift values than usually assumed, being at $z \approx 0.1$ if a 25% - 30% error margin in the measurements of this density are taken into consideration. Such a departure from apparent homogeneity occurs well within the region of validity of the Hubble law, which is a first order linear approximation for the redshift-distance relation in this cosmology. This happens because the approximation of local homogeneity in this model is a zeroth order approximation, while the Hubble law is a first order one.

By studying the behaviour of the correlation function along the past null geodesic, the results indicate that it does not seem able to fully characterise the density distribution of the model, being in fact a poor statistical probe for the behaviour of the average density of the model, while Pietronero's (1987) conditional average density seems to fulfill this task as a statistical test which can be directly derived from galaxy catalogues. On the other hand, Pietronero-Wertz conditional density seems unable to characterise quantitatively the departure from apparent homogeneity of the model, although it will give a qualitative indication of such an effect. Both functions, however, will basically give the same statistical information within our local region.

Finally, these results were obtained by using the method of expanding the relativistic observational relations in power series, a method in the spirit of the pioneering work of Kristian & Sachs (1966).

The results of this paper can be interpreted in a variety of ways. First of all, they do not disprove the Friedmann models in the sense that we may still be observing galaxies in our present time hypersurface, and so the homogeneous region could still be within reach.⁶ However, this possibility can only be realised within the Friedmann models if either H_0 or Ω_0 or both are smaller than the values used in this paper. For instance, as we know that in open Friedmann models the scales where the observable homogeneity finishes is deeper than in the Einstein-de Sitter cosmology (Ribeiro 1993), we can envisage an open model with a value of Ω_0 such that the observable homogeneous region ends at much deeper redshift values than $z = 0.1$, as opposed to the case of the $\Omega_0 = 1$ model studied in this paper. If H_0 is found to be in the lowest limits of its present uncertainty, the observable homogeneous region could still be pushed to even further depths. A closed model seems at first very difficult to

⁶ Strictly speaking, we never observe galaxies in our present time hypersurface. Consequently, what we really do is an "effective observation" at our hypersurface, which is always limited to the error margins. If observations were perfect, without errors, we would never be able to do calculations and data reduction as if there were no past light cone problem, but since observations do have error margins we can approximate and consider an effective observation at the present time hypersurface provided we always keep in mind that this is an approximation valid only within the error margins considered.

be reconciled with the presently available astronomical data as it would bring such scales to much closer distances, even closer than the ones shown in figure 1 for the Einstein-de Sitter model. Therefore, the only Friedmann models that seem able to be reconciled with current expectations of an yet to be reached homogeneity are the ones with low Ω_0 and H_0 , and this would favor open models in such scenarios.⁷

These results also seem to make a fractal universe a more attractive possibility (Pietronero 1987; Ruffini, Song & Taraglio 1988; Coleman & Pietronero 1992; Ribeiro 1993, 1994) since at small scales a fractal description of the observed lumpiness of matter seems to be a quite good approximation (Pietronero 1987; Peebles 1989), and as at deeper ranges we start to observe a deviation from apparent homogeneity in Friedmann models, we can envisage a single fractal or multifractal pattern going further, up to the present scales of observations, or, perhaps, even beyond.

On the problem of relativistic effects in observational cosmology, the current practice is such that it implicitly leads to a situation where the assumption that relativistic effects can be ignored in the study of galaxy correlation and redshift surveys remains basically untested. Nevertheless, from the results given above, it seems reasonable to say that we cannot ignore space curvature and expansion in cosmology at close ranges where it has been usually thought this can be done without penalty (Peebles 1980, p. 143). It is therefore clear that relativistic effects start to play a role at the first order approximation of at least some observational relations, and we can only disregard such effects at the zeroth order of approximation. In other words, the Hubble law is basically a first order approximation while an Euclidean treatment of cosmology is good enough only up to zeroth order of approximation.⁸ This is an important point since the usual thinking implicitly assumes that the first order approximation is well enough for an Euclidean treatment in cosmology, and the analysis of this paper shows that such an assumption may lead to wrong results and misleading interpretation because current reduction and analysis of data obtained from not so deep surveys may be subject to the real possibility of systematic errors. Therefore, the statement “at small redshifts”, widely encountered in the literature as implicitly meaning that “relativistic effects can be ignored at these scales”, should be used with much greater care.

Finally, from the point of view of a statistical analysis of data, since figures 1 and 3 show the theoretical form of the number density distribution in the Einstein-de Sitter model, by accessing the astronomical data we could in principle compare the theoretical predictions with observations in order to see if this suffice to prove or disprove the hypothesis of spatial homogeneity of this model. This way of looking at the results of figures 1 and 3 has the advantage of making the concept of spatial homogeneity testable as Γ and Γ^* are better descriptors of the average density than ξ , and could provide a sort of theoretical fingerprint

⁷ To summarize, if $H_0 > 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$, the curves of figures 1 and 3 are contracted. If $H_0 < 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$, the same curves are stretched.

⁸ By Euclidean treatment of cosmology I mean the use of common Euclidean geometry and the assumption that the dynamical theory for light given by the theory of relativity can be ignored.

for the Einstein-de Sitter model which can be tested observationally.

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Table Captions

Table 1: Simple numerical estimates showing that the deviations from the linearity of the redshift-distance relation seem to be a second order effect, while the departures from the local spatial homogeneity are a first order effect. The figures show that in the Einstein-de Sitter model with $H_0 = 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$, relativistic effects would start to matter at $z \approx 0.04$, which is also the range where the zeroth order locally homogeneous approximation would start to be no longer valid.

Table 2: Comparison between the percentage drop from the 100% value at $t = 0$, for $\langle n \rangle$ and Γ . Considering that the average number density gives the actual distribution of dust of the Einstein-de Sitter model, the conditional density (third column) does not seem to be much sensitive to the drop of the density itself, although it does indicate such decrease qualitatively.

Figure Captions

Fig. 1: Plot of average number density $\langle n \rangle$ vs. the redshift z where the value $H_0 = 75 \text{ Km s}^{-1}\text{Mpc}^{-1}$ for the Hubble constant was assumed. This figure is similar to the one presented in Ribeiro (1992) and shows the drop in the density with the redshift in the Einstein-de Sitter model. Note that this drop becomes significant even when $z \ll 1$, although at this range the redshift-distance law is well approximated by a linear function.

Fig. 2: The correlation function $\xi(d_\ell)$ plotted for different sample sizes R . The graph indicates that for each different value of R the correlation function behaves as if it had obtained a different scale length for the breakdown of the observable local homogeneity, in contrast to the values of table 1 for the real distribution of dust.

Fig. 3: The conditional density $\Gamma(d_\ell)$ for the distribution of dust in the Einstein-de Sitter model (eq. [15]), plotted in the same range as in the case of the average number density of figure 1. One can see that this function traces qualitatively the tendency of the dust of deviating from local homogeneity as the distance increases (shown in figure 1).

Table 1

z	d_ℓ (Mpc)	a	b
0.003	12	99%	0.08%
0.01	40	97%	0.2%
0.036	145	90%	0.9%
0.07	285	82%	1.7%
0.1	410	75%	2.4%
0.125	515	70%	3%
0.5	2202	30%	11%

^aPercentage drop of $\langle n \rangle$ from the 100% value at $t = 0$ (1st order effect).

^bPercentage contribution of the 2nd order term in the redshift-distance law (2nd order effect).

Table 2

z	a	b
0.003	99%	99.9%
0.01	97%	99.8%
0.036	90%	99.1%
0.07	82%	98.3%
0.1	75%	97.5%
0.5	30%	87.7%

^aPercentage drop of $\langle n \rangle$ from the 100% value at the present time hypersurface $t = 0$.

^bPercentage drop in $\Gamma(d_\ell)$ from the 100% value also at $t = 0$.

Figure 1

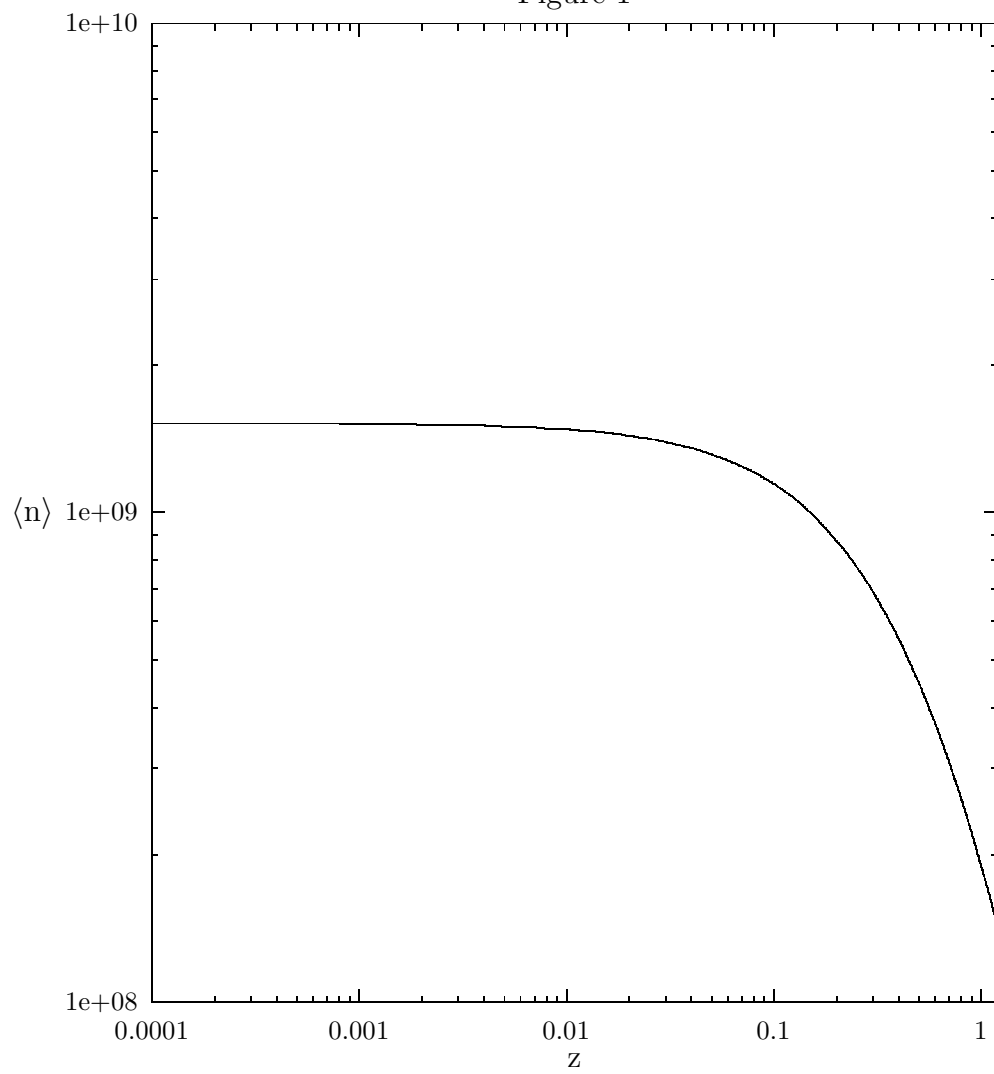


Figure 2

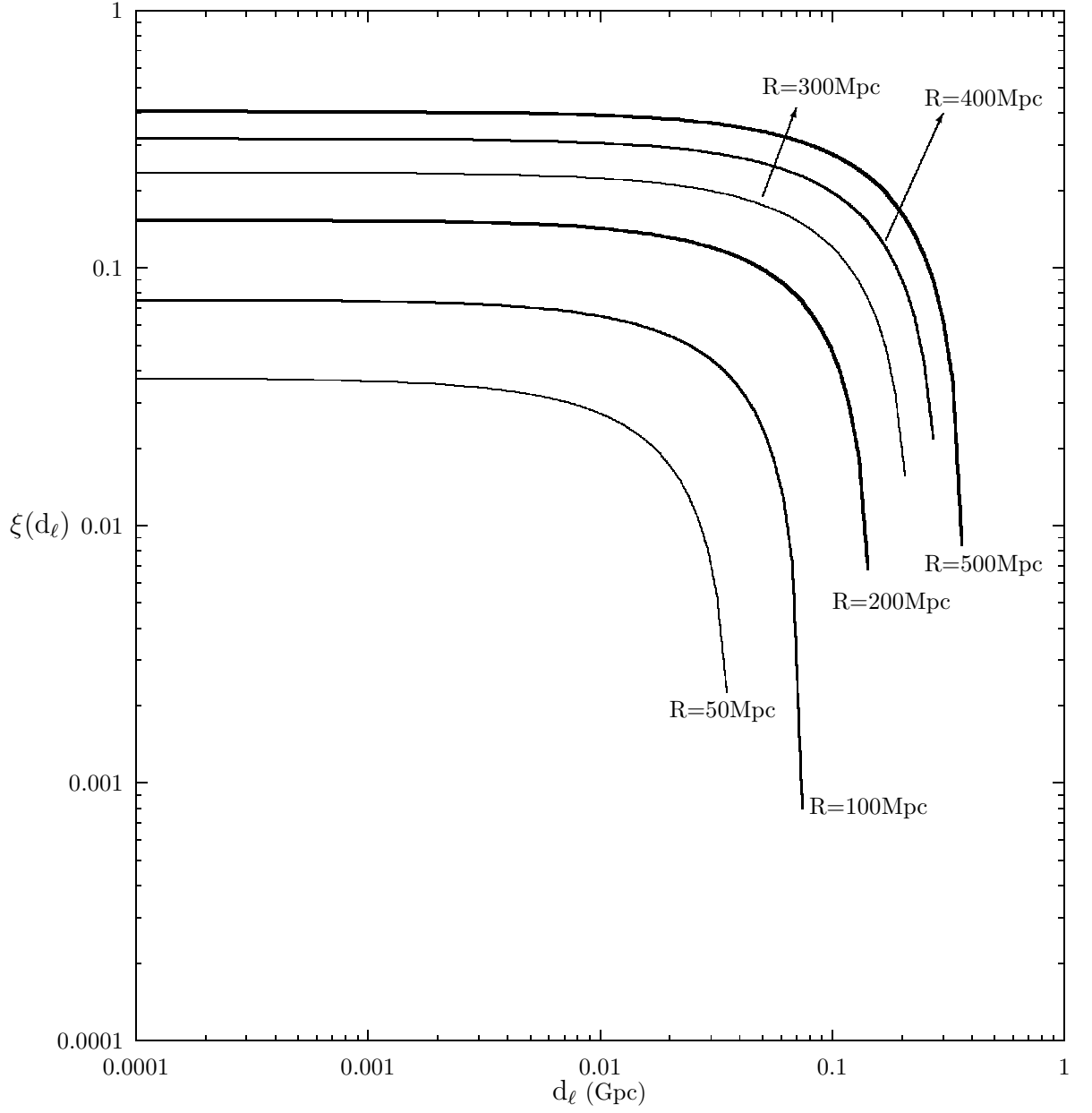


Figure 3

